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Manifolds and Group Actions

Homework 1

Mandatory Exercise 1. (10 points)
Consider the following topological spaces.

(1) The 2-**sphere** $S^2 := \{x \in \mathbb{R}^3 \, : \, |x|^2 = 1\} \subset \mathbb{R}^3$

(2) The **real projective plane** $\mathbb{R}P^2 := \mathbb{R}^3 \setminus \{0\}/_\sim$, where $x \sim y$ if and only if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $x = \lambda y$.

(3) The **open** 3-**ball** $D^3 := \{x \in \mathbb{R}^3 \, : \, |x| < 1\}$

(4) The **closed** 3-**ball** $\overline{D^3} := \{x \in \mathbb{R}^3 \, : \, |x| \leq 1\}$

(5) $S^2/_\sim$, where $x \sim y$ if and only if $x = y$ or $x = -y$.

(6) $\mathbb{R}^3$

(7) The 3-**Torus** $S^1 \times S^1 \times S^1$, where S^1 is defined in Exercise 2.

(8) $\{(x, y) \in \mathbb{R}^2 \, : \, xy = 0\}$

(9) $\{(x, y, z) \in \mathbb{R}^3 \, : \, x^2 + y^2 = z^2\}$

(a) Make sketches of all sets.

(b) Which of these spaces are topological manifolds and which not? Give a proof only for one (of the non-trivial) sets which is a manifold and for one set which is not.

(c) Which of these manifolds are homeomorphic and which are not homeomorphic?

Mandatory Exercise 2. (10 points)
Consider the n-sphere

$$S^n := \{x \in \mathbb{R}^{n+1} \, : \, |x|^2 = 1\} \subset \mathbb{R}^{n+1}$$

with the subspace topology and let $N = (0, \cdots, 0, 1)$ and $S = (0, \cdots, 0, -1)$ be the north and south poles. The **stereographic projection** from N is the map

$$\pi_N \colon S^n \setminus \{N\} \longrightarrow \mathbb{R}^n$$

which takes a point p on $S^n \setminus \{N\}$ to the intersection point of the line through N and p with the hyperplane $\{x_{n+1} = 0\} \subset \mathbb{R}^{n+1}$. And analogous (with N replaced by S) the stereographic projection from S.

(a) Make sketches of both projections and give explicit formulas for them.

(b) Check that $\{(\mathbb{R}^n, \pi_N^{-1}), (\mathbb{R}^n, \pi_S^{-1})\}$ is an atlas for S^n.

The maximal atlas obtained from this is called the **standard differentiable structure** on S^n.

Suggested Exercise 1. (0 points)
The **Möbius strip** M is obtained from the square $I \times I$ with $I = [-1, 1]$ by identifying points $(x, -1)$ with points $(-x, 1)$.

(a) Make a sketch of M and show that M is a topological manifold with boundary.

(b) Show that the space obtained by gluing together the Möbius strip M and a closed 2-disk $\overline{D^2}$ along their boundaries is again a topological manifold.

(c) Show that this manifold is homeomorphic to the real projective space $\mathbb{R}P^2$.

Suggested Exercise 2. (0 points)
A **triangulation** of a 2 dimensional topological manifold M is a decomposition of M in a finite number of triangles (i.e. subsets homeomorphic to triangles in $\mathbb{R}^2$) such that the intersection of any two triangles is either a common edge, a common vertex or empty (it is possible to prove that such a triangulation always exists). The **Euler characteristic** of M is

$$\chi(M) := V - E + F,$$

where V, E, F are the number of vertices, edges, and faces of a given triangulation of M (it can be shown that this is well defined, i.e. does not depend on the explicit choice of the triangulation of M).

(a) Show that adding a vertex to a triangulation does not change $\chi(M)$.

(b) Compute the Euler characteristic of S^2, $\mathbb{R}P^2$, of the closed disk $\overline{D^2}$, the **annulus** $S^1 \times I$, and of the Möbius strip.

(c) Show that the 2-**torus** $T^2 := S^1 \times S^1$ and S^2 are not homeomorphic.

Suggested Exercise 3. (0 points)
Let M be the disjoint union of $\mathbb{R}$ with a point p and consider the maps $f_i \colon \mathbb{R} \to M$ for $i = 1, 2$ defined by $f_i(x) = x$ if $x \in \mathbb{R} \setminus \{0\}$, $f_1(0) = 0$ and $f_2(0) = p$.

(a) Show that $f_i^{-1} \circ f_j$ are differentiable on their domains.

(b) If we consider the atlas formed by $\{(\mathbb{R}, f_1), (\mathbb{R}, f_2)\}$, then the corresponding topology will not satisfy the Hausdorff axiom.

Hand in: Tuesday April 25th
in the first exercise session